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A Market Consistent Approach to the Valuation of No Negative Equity Guarantees and Equity Release Mortgages

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Abstract

This paper provides a new market consistent approach to the valuation of No Negative Equity Guarantees and Equity Release Mortgages. The paper innovates in two respects. First, it provides a new treatment of net rental yields and deferment rates and a proof that the two are equal. Second, the paper provides a new approach to the estimation of the volatility inputs. The proposed approach to volatility produces a volatility term structure that is dependent on the age and gender of the borrower. Illustrative valuations are provided based on the Black '76 put pricing formula and mortality projections based on the M5 Cairns-Blake-Dowd (CBD) mortality model. Results have interesting ramifications for industry practice and prudential regulation.

Keywords: Black '76 model, CBD mortality model, Equity Release, No Negative Equity Guarantee

JEL Classification: G2, G3

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1. Introduction

An ERM is a loan made to a property-owning borrower late in life that is collateralised by the value of their property. The amount of the loan compounds over time at the loan interest rate and the loan is repaid when the borrower leaves their property by dying or going into care. In the UK, ERMs typically include No-Negative Equity Guarantees (NNEGs) which stipulate that the amount of the loan due for repayment is capped by the property value at the time the loan is repaid, i.e., the borrower or their estate will owe no more than the minimum of the rolled-up loan amount and the property value at the time of repayment. This obligation to repay the minimum of two future values implies that the NNEG involves put options granted by the lender to the borrower.

The question then is how to value the NNEG guarantee. We propose a 'market consistent' (MC) valuation approach as that term is understood in actuarial circles.[1] We can define an MC valuation as a 'fair value' valuation based on the International Financial Report Standards (IFRS) definition of a fair value price, namely

> The price that would be received to sell an asset or paid to transfer a liability in an orderly transaction between market participants at the measurement date[2]

An alternative (and for our purposes practically equivalent) definition is that provided by Tim Gordon: he defines a market consistent valuation as one which is consistent with modern finance theory as the term is used in Exley, Mehta and Smith (1997).[3] More precisely, we use a MC approach that combines the Black '76 put pricing model and the M5 CBD mortality model (Cairns et al. 2006, 2009).

In the UK context, the earlier NNEG literature includes Hosty et al. (2008), Li et al. (2010), Dowd (2018), Jeffery and Smith (2019), Buckner and Dowd (2020a on which the present paper draws), Dowd et al. (2019) and a series of regulatory documents set out by the UK Prudential Regulatory Authority (see e.g., PRA, 2016, 2018).

The present paper contributes to the literature on NNEG and ERM valuation in two respects. First, it establishes that the deferment rate, i.e., the discount rate applied to the spot property price to obtain the present value of the forward property price, is equal to the net rental yield on the property. Second, it provides a new analysis of the volatility inputs (note the plural) to the put option valuation formula that is used to obtain the NNEG valuation. It turns out that there is no single 'one size fits all' volatility input that applies to borrowers of all ages, as

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[1] For more on market consistent valuation see, e.g., Malamud et alia (2008) or Wüthrich (2016).
[2] See, e.g., https://www.iasplus.com/en/standards/ifrs/ifrs13.
[3] See Gordon (1999) and Exley et al. (1997).

had hitherto been supposed. Instead, there is a term structure of volatility inputs spanning the borrower's possible future lifetime, and their gender too.

This article is organised as follows. Section 2 sets out the basics of NNEG and ERM valuation. Section 3 discusses net rental and deferment rates. Section 4 discusses the volatility inputs to the option pricing equation. Section 5 provides some illustrative results and section 6 concludes.

2. The Basics of NNEG and ERM Valuation

The NNEG valuation model has two key ingredients: a set of expected house-exit probabilities and a put option pricing model.

Exit probabilities

The house exit probabilities refer to the probabilities that the borrower will exit the house (and hence terminate the loan) over each of the next 1, 2, 3, … etc years. We assume away complicating factors such the possibility of early repayment of the loan and the possibility that the individual will spend time in a care home at the end of his or her life.[4] We assume instead that the individual will die at home, i.e., so house exit occurs at death. Under these assumptions, the exit probability for year t is equal to

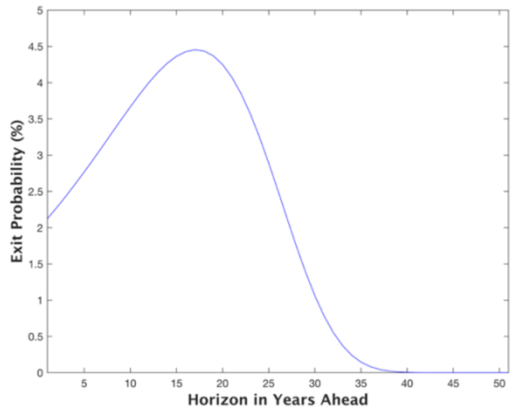
(1) $$exit\ prob_t = q_t \times S_t$$

where q_t is the mortality rate for year t and S_t is the probability that an individual alive now will survive to year t. Note that $S_0 = 1$ and $S_t = (1 - q_{t-1})S_{t-1}$ for all $t > 0$.[5]

Figure 1 House Exit Probabilities for Males Currently Aged 70

[4] For more on early repayment and long-term care, see Buckner and Dowd (2020a, chapters 15 and 11) respectively.

[5] The 'q' terminology for the mortality rate is standard in the mortality literature and a little unfortunate in the NNEG context, where q is typically used to refer to the deferment rate. The reader should bear this ambiguity in mind, but the context should make it clear whether it is the mortality rate or the deferment rate that we are referring to, and it is mostly the latter.



Notes: House exit probabilities are based on CBD-M5 model (Cairns *et alia*, 2006, 2009) cohort mortality rate projections using male England & Wales deaths rate data estimated over ages 55:89 and years 1971:2017. Source: Life & Longevity Markets Association.

The left hand (low t) house exit probabilities are close to the low t mortality rates and reflect the early high survival probabilities (i.e., that people aged 70 have a high probability of living at least a few years), and the later (high t) house exit probabilities primarily reflect the fact that the probabilities of living to extreme old age are low and approach zero in the limit.

Valuation Issues and the Put Pricing Model

The present value ERM of the Equity Release Mortgage loan can be considered to be the present value L of a risk-free loan, one which is guaranteed to be repaid in full, minus the present value $NNEG$ of the NNEG guarantee:

(2) $$ERM = L - NNEG$$

The original loan amount grows at the loan rate (sometimes called the rollup rate) l from its current amount until the time when the loan ends. Therefore L is given by

(3) $$L = \sum_t [exit\ prob_t \times current\ loan\ amount \times e^{(l-r)t}]$$

where $exit\ prob_t$ is the probability of exiting the house in period t and r is the risk-free interest rate.[6]

The valuation of L is straightforward.

[6] Note the implicit distinction here between the loan amount or rolled up loan amount, on the one hand, and L, the (economic) value of the loan, on the other. The former is the amount loaned plus the interest accumulated since the inception of the loan, whereas the latter is the value of the loan to the lender, including the expected profit on the loan. A concrete example of the distinction between the two is given in Table 1. Note too that the economic value of the loan is not to be confused with the accounting book value of the loan, which is another issue again.

$NNEG$ is the sum of the products of the exit probabilities for each future time t and the present value of the NNEG guarantee for each future time t:

(4) $NNEG = \sum_t [exit\ prob_t \times NNEG_t]$

where $NNEG_t$ is the present value of the NNEG guarantee for period t.

The question is then how to value each of these individual $NNEG_t$ 'nneglet' terms.

Recall that the NNEG gives the customer (or the person acting for the customer) the right to repay the loan by paying the lender the minimum of the loan value or the house price at the time of death.

The right to repay the minimum of two future values (one of which, the future house price, is uncertain) at some given future time implies a European put option granted by the lender to the borrower. Since the time of exercise is uncertain, we can think of the NNEG as involving a portfolio of such put options.

In the case of our put options the underlying variable is a residential property ('house') or more precisely, a forward contract on a house, and we should think of a house as an asset that bears a continuous yield in the form of a net rental yield. This net rental yield reflects the use benefit of living in the house or the (net, after costs) rental income we might get by renting the house out.

A natural option pricing model to use in these circumstances is the Black '76 model (Black, 1976). Black '76 is an appropriate pricing model when the underlying is a forward contract with a maturity coterminous with that of the option itself. This model is a near-relative of the famous Black-Scholes-Merton model (Black and Scholes, 1973; Merton, 1973).[7]

The Black '76 formula for the price p_t of a European put option with maturity t on a forward contract on a commodity bearing a continuous yield q is given by the formula:

(5) $p_t = e^{-rt}[K_t N(-d_2) - F_t N(-d_1)]$

where r is the risk-free rate of interest, K_t is the strike or exercise price for period t, F_t is the forward house price for period t, the function $N(\dots)$ is the value of the cumulative standard normal distribution at the value specified in brackets, and d_1 and d_2 are given by:

[7] In principle, we can also use the BSM model if we allow the underlying to bear a continuous dividend yield, which will be the net rental yield. The equivalence of the two models under these circumstances is shown in Buckner and Dowd (2020a, appendix to chapter 3).

(6) $d_1 = [ln(F_t/K_t) + \sigma_t^2 t/2]/(\sigma\sqrt{t})$

(7) $d_2 = d_1 - \sigma_t\sqrt{t}$

where σ_t is the volatility of the forward house price for maturity t.

The strike price K_t is then the rolled up or accumulated loan amount by period t:

(8) $K_t = current\ loan\ amount\ \times e^{lt}$

and the forward price F_t, the price agreed now to be paid on possession in period t, is:

(9) $F_t = current\ house\ price\ \times e^{(r-q)t}$

where q is the *deferment rate*, namely the discount rate applied to the current house price to give the *deferment price*, the price we would agree to pay *today* to take possession of the house in t years' time. Thus, the deferment house price R_t is given by:

(10) $R_t = current\ house\ price\ \times e^{-qt}$

The difference between the forward house contract and the deferment house contract is that with the forward we settle when we take possession *in t years' time*, but with the deferment contract we settle *today*.[8] Therefore, the deferment house price R_t is the present value of the forward price, where the present value is obtained by discounting at the risk-free rate r.

It is important to note that the deferment house price will be less than the current house price S_0 because the deferment rate $q > 0$.

The forward house price F_t should not be confused with future house prices or expected future house prices:

- Forward prices for future period t are known (or can be approximated) now and we need to be able to price options using information available now.
- Options cannot be priced using future house prices because future house prices are currently unknown.
- Expectations of future prices do not appear in the Black '76 option pricing formula.

[8] See PRA SS 3/17 (p. 12, note 2).

We should also keep in mind that although the original Black '76 article discussed options on futures, futures prices are the prices of *futures* contracts, a form of forward contract, *not* actual or expected *future* prices of any sort.

A mistake to be particularly avoided – the one common among UK ERM actuaries – is to confuse forward and expected future prices. This mistake typically manifests itself in the inputting of an assumed expected house price inflation rate into (9) instead of the forward rate $r - q$.

To repeat: it is not the *future* or expected future price of a contract for *immediate* possession that we use in the option pricing equation, but rather the *current* price of a contract for *future* possession.

3. Net Rental Yield and Deferment Rate

Let's start by some clarifying definitions. The net rental yield is the amount that the landlord receives after deducting for void, management costs and maintenance costs, divided by the property price. The deferment rate, as defined by the PRA, is "the discount rate that applies to the spot price of an asset resulting in the deferment price, where the deferment price is the price that would be agreed and settled today to take ownership of the asset at some point in the future".[9] The deferment price reflects the foregone income or use during the deferment period.

These two rates are frequently run together, and as we shall show shortly, they are in fact mathematically identical, but they are defined differently.

Proof that the Deferment Rate Equals the Net Rental yield

From the above definition, the deferment rate is the discount rate that when applied to the freehold price of vacant possession results in the price of deferred possession. The deferment rate itself is not directly market observable, but it can be estimated as a function of market variables. The method proposed here uses net rental yields, as follows.

Let d be the *current* net nominal annual rental, the current time being the *beginning* of the year. (We use 'd' here because the approach we are using derives from the dividend discount model, where 'd' is used to refer to (nominal) dividends.) 'Net' means the gross or headline rental paid by tenants, less the costs incurred by the lessor such as management, maintenance and the expected costs of void or empty periods while the property is being re-let. Then we shall show that

[9] PRA SS 3/17, July 2018.

(11) $q = d/S$

where: q is the deferment rate, d is the net rental amount as above, with the current time being the beginning of the rental year; and S the estimated 'spot price', i.e., the freehold value of vacant possession estimated as the market value of an identical or similar property not encumbered by a leasehold.

Assume that the value S of a perpetual income producing asset is the sum of the present values of its individual cashflows. (This assumption is the bedrock of practically all financial theory.[10]) Assume that the discount factor is given by $1/(1 + r + \pi + \lambda)$, where r is the risk free rate, π the risk premium demanded by investors for taking on risky cashflows, and λ the illiquidity premium demanded by investors for cashflows that cannot easily be exchanged at market. Assume also that cashflows grow at a constant rate g, so that future cashflows are $d(1 + g)$, $d(1 + g)^2$, etc. Define y as follows

(12) $1/(1 + y) = (1 + g)/(1 + r + \pi + \lambda)$.

Then from the definitions and assumptions above

(13) $S = d[1/(1 + y) + 1/(1 + y)^2 + 1/(1 + y)^3 \dots]$

and from the definition of the deferment price R_n as the present value of the future cashflows *minus* the present value of the first n cashflows, i.e. minus the "foregone income"

(14) $R_n = d[1/(1 + y)^{n+1} + 1/(1 + y)^{n+2} + 1/(1 + y)^{n+3} \dots]$

Divide every term in (14) by $(1 + y)^n$, express in terms of S, and note that y meets the definition of q as the discount rate we apply to the 'spot' price S to give the deferment price R_n.

(15)
$$\begin{aligned} R_n &= d[1/(1 + y) + 1/(1 + y)^2 + 1/(1 + y)^3 \dots]/(1 + y)^n \\ &= S/(1 + y)^n \\ &= S/(1 + q)^n. \end{aligned}$$

Define X as S/d. Then, substituting X into (13), and substituting q for y

(16) $X = 1/(1 + q) + 1/(1 + q)^2 + 1/(1 + q)^3 + \dots$.

Multiply both sides by $(1 + q)$

[10] It underlies, for example, in the Discount Dividend Model (e.g., Gordon, 1959) with property prices and rentals taking the place of stock prices and dividends. See, e.g., https://en.wikipedia.org/wiki/Dividend_discount_model (accessed 25 June 2020).

(17) $X(1+q) = 1 + 1/(1+q) + 1/(1+q)^2 + ...$
$= 1 + X.$

Divide both sides by X and subtract 1, substitute S/d for X and rearrange:

(18) $q = 1/X = d/S$

which was to be proved.

Observe that (18) holds whatever is on the right hand side of (12), i.e., it holds whatever constant growth rate we choose, and whatever the interest rate, risk premium or illiquidity premium that might be required by investors.

To give an example, Sheffield City Council recently reported that the average house price in the city is £163,288 and the average private monthly house rental is £600.[11] The corresponding average annual house rental is £600 times 12 = £7,200. We then have to make a judgment about the relationship of net rental to gross and let's suppose that net is 75% of gross (but see also the next subsection for more on this breakdown of gross to net). The average net rental would then be 75% times £7,200 = £5,400, giving us a q = 5,400/163,288 = 3.3%.

One can then envisage equity release lenders a similar approach to drill down further to obtain q calibrations for neighbourhood, property class and even individual properties.

Calibrating the Net Rental Yield

We can provide a more precise calibration of the net rental yield.

We first decompose the net rental yield as follows:

(19) $net\ rental\ yield = y - v - c - m$

where y is the gross rental yield or the yield paid by the tenant, v is the void rate, c is management cost and m is the maintenance cost.

Define the maintenance cost m as the rate of expenditure (as a percentage of gross rental) required to keep the property in perfect condition (i.e. such as to achieve the best sales price for a property of that size in the same area), and define the tenant maintenance share (s) as the proportion of m that the tenant is likely to spend on maintenance. s will typically vary between 0 and 100%. For a short rental, s will be close to zero, and for a long let we would expect s to be close to 100% in the early years of tenancy, falling over time. In the final years it

[11] *Sheffield Housing Market Bulletin*, January-March 2019, Sheffield City Council.

might fall to zero, even for a standard tenancy, given the lack of incentive to keep in full order for the landlord's benefit.[12] For an ERM, it would seem unlikely that the 'tenant' at end of life, perhaps in the situation where the NNEG had bitten, would have any incentive to keep the property in good condition, so we would expect s to fall toward zero in that case too.

We now use the following calibrations:

- Void: we use the standard '1 month in 12' rule of thumb, i.e., $v = (1/12) \times y$.[13]
- Management cost: following Buckner and Dowd (2020, p. 37), we assume management cost $c = 10\% \times y$.
- Maintenance cost: again following Buckner and Dowd (2020, p. 37), we assume maintenance costs $m = 15\% \times y$.

Thus, the maintenance cost borne by the landlord and to be subtracted from the gross rental yield is $m = 15\% \times 50\% \times y = 7.5\% \times y$.

We then have

(20) $$net\ rental\ yield = y \times (11/12 - 0.1 - 0.075) = y \times 0.7417.$$

Thus, the net is 74.17% of the gross.

Again following Buckner and Dowd (2020, p. 37), we take

(21) $$y = 5.6\%.$$

Therefore

(22) $$q = net\ rental\ yield = 74.17\% \times 5.6\% = 4.15\% \approx 4.2\%.$$

So we use $q = 4.2\%$ as our 'best estimate' of the deferment rate.[14,15]

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[12] In fact, we can also imagine $s < 0$. So if $s = 0$ reflects no active effort to keep the property in condition, $s < 0$ reflects a determined effort by occupiers to strip the property (e.g., of light fittings, marble fireplaces, etc.) or trash the property! It happens.

[13] An alternative is to use empirical void data. Average void period for landlords in private rented sector in the United Kingdom (UK) have varied from 2.4 weeks to 2.9 weeks. (Source: https://www.statista.com/statistics/421102/rental-properties-void-periods-in-the-uk/. Accessed 19 March 2019.) If we take the mid-point, 2.65 weeks, then the average void rate by this measure would be 2.65/52 = 5.1%, as compared to the 'rule of thumb' void rate of 11/12 = 8.3%.

[14] If we use the empirical void rate of 5.1%, then net is 77.4%% of gross and we would obtain $q = 4.3\%$.

[15] We have implicitly assumed that the deferment rate is constant over the term to maturity, but the analysis can easily be tweaked to accommodate a term structure for the projected deferment rate.

4. Volatility

The standard approach to volatility estimation takes the volatility to be the standard deviation of the return to the underlying, where the latter is often taken to be a house price index (HPI) and some adjustment is made for the impact of autocorrelation in the HPI. A good example is CP 13/18 (p. 9), which states:

> 2.16 The PRA estimated a value for the property volatility parameter from analysis of residential property price index data. Nationwide, Halifax and Office for National Statistics index data were analysed and several time series models were fitted to the quarterly log-returns of data sets over a variety of historical time periods. The PRA selected a parsimonious model that fitted the data well, and extracted from the model the unconditional volatility for various holding periods, allowing for autocorrelation. Further adjustments were made to allow for concentration risk and basis risk between the changes in prices of individual properties and the index. The PRA's *central estimate is of a 13% volatility assumption* for typical holding periods for ERMs, although use of alternative data choices gives a range of 13%-16%, and making an allowance for parameter uncertainty gives a range of 11%-18%. Estimates for property volatility provided to the PRA by firms are generally in the *range 10%-15%*. (Our emphasis)

We would suggest that this approach is unsatisfactory for two reasons. First, it implicitly supposes that there is one single 'one size fits all' volatility input to NNEG valuation or, more specifically, that each of the put 'nneglets' that make up the NNEG should have the same numerical volatility input. We suggest that this implicit assumption is mistaken, and that each nneglet should in principle have a different volatility input. Second, it is not necessary,[16] and not advisable,[17] to adjust for autocorrelation in the underlying.

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[16] That it is not necessary to take account of autocorrelation in the underlying has already been established by Cornalba et al. (2002). They provide a fairly general analysis of the impact of temporal (i.e., auto-correlation) on option pricing and their conclusions are clear. "In the Gaussian case [the one considered in Black-Scholes], we find that the effect of [auto-] correlations can be compensated by a change in the hedging strategy and therefore *options should be priced using the standard uncorrelated Black-Scholes model* (our italics)."

[17] Suppose one estimates the volatility correctly ignoring autocorrelation. One then determines that the underlying is autocorrelated and adjusts the volatility to 'take account' of the autocorrelation. Then the new estimate must be wrong, because the first one was already correct.

We need to go back to first principles. We propose a new approach that starts with the volatility of the forward return derived from a house price index (HPI) and then gradually adds in further risk factors.

Vol Estimation: A First Pass

We begin by defining the volatility σ of the return on a forward contract as the square root of the variance of that return:

(23) $\sigma^2 = \frac{1}{n}\sum_1^n (FR_t - \overline{FR})^2.$

where FR_t is the forward return over the period to time t.

More specifically, we seek to obtain the variance of the return on a forward with maturity T, $F_{t,T}$ say, based on observations of a spot price S_t and we would typically take S_t to be some HPI. We already know (see (9) above) that

(24) $F_{t,T} = S_t e^{(r-q)T}$

where the variables have their usual interpretations.

The return on a forward with maturity T is

(25) $FR_{t,T} = lnF_{t,T} - lnF_{t-1,T}$

where variables have their obvious interpretations. Substituting (24) into (25)

(26)
$$\begin{aligned} FR_{t,T} &= \ln\,[S_t e^{(r-q)T}] - \ln\,[S_{t-1} e^{(r-q)T}] \\ &= \ln S_t + (r-q)T - \ln S_{t-1} - (r-q)T \\ &= \ln S_t - \ln S_{t-1} \end{aligned}$$

which is the return on the spot house price.

Therefore, the variance of $FR_{t,T}$ must be equal to the variance of the spot return, regardless of the maturity of the forward.

So what should we take the value of the spot return variance to be?

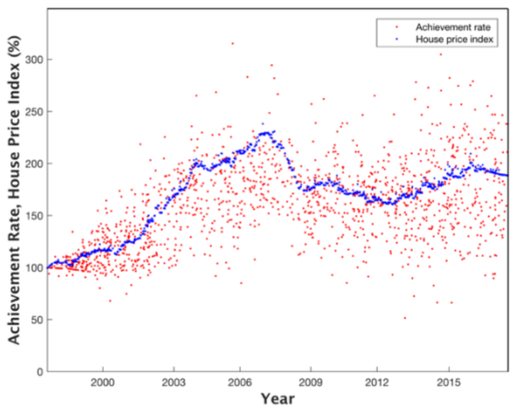
We would suggest to go with the PRA's central estimate of 13%.

Volatility Around the Index

The estimated 13% volatility only refers to the volatility of the index, but there is also the volatility around the index. This additional volatility would include the impact of regional variation around the index, but there are further contributory factors as well. These include, e.g., the impact of changes in consumer relative demand for different types of property, expansions of nearby roads, the impact of new housing estates, yuppification, middle class flight, the opening or closing of a good nearby school, and so forth.

The next Figure shows scatterplot of the 'achievement rates' of ERMed properties, i.e., the amounts that the lender was able to realise after the borrower exited, expressed as a percentage of the indexed value, based on the Shared Appreciation Mortgage Securities (SAMS) originated by HBOS in the late 1990s:

Figure 2: Indexed vs. Achieved House Prices



Source: SAMS

The red dots are the scattershot of the individual achievement rates in the sample. The darker blue random-looking line is a simulated house price index.

We immediately see that the achieved values are much more volatile than the index. Above all, when seeking to calibrate the volatility, we need to keep in mind that it is that dispersion that matters, not the volatility of the index itself.

It would then behove us to revise our earlier index-based volatility estimates to take account of this additional source of volatility. We could start with our earlier index volatility. Let us label this volatility as σ_{INDEX}. We now obtain σ_{AR}, the volatility of the achievement rate, as follows: take a rolling standard deviation of the achievement rate and divide by root time to get an annualised value. We found that the annualised volatility values vary from 7% to 10%. Let's work with the middle value of 8.5%. We then have to assume a plausible correlation between the index vol and the achievement rate vol. Assuming zero correlation, which is not unreasonable, we obtain the results reported in Table 1:

Table 1: Illustrative Volatility Including Impact of Achievement Rate Volatility

σ_{INDEX}	σ_{AR}	$\sigma_{INDEX\ and\ AR}$
13%	8.5%	15.5%

Note: The term in the rightmost cell are obtained by Pythagoras.

Interest Rate Risk as a Further Contributor to Volatility

We have hitherto assumed (as per Black-Scholes/Black '76) that the interest rate is constant, i.e., that there is no interest rate risk. In fact, interest rate risk not only exists, but arises from two different sources. Recall the following components of the Black model, reproduced here in slightly simplified form:

$$p_t = e^{-rt}[K_T N(-d_2) - F_T N(-d_1)] \quad (27)$$

$$d_1 = [ln(F_T/K_T) + \sigma_T^2 T/2]/(\sigma_t T) \quad (28)$$

$$d_2 = d_1 - \sigma_T \sqrt{T} \quad (29)$$

$$F_T = S\, e^{(r-q)T} \quad (30)$$

where p_t is the value of the t decrement put, K_T the strike price, F_T the forward price, σ_T the annualised input volatility, T the time to maturity in years, r the interest rate, S the price of 'spot' possession of the property, and $N(\ldots)$ is the cumulative normal distribution function.

The interest rate term r appears first (see (27)) as a *discount term* wrapped around the terms representing the future value of the put option, which brings the future value (i.e., $[K_T N(-d_2) - F_T N(-d_1)]$) back to present value. Here r plays the role of an outer discount factor.

r then appears again (see 30)) as a *projection term* or inner discount factor taking us from the spot price S to the forward price F_T.

Each appearance gives rise to interest rate risk, but in different ways.

Discount rate risk

The first can be called discount interest rate risk. This risk can be hedged relatively easily, the gist of it being to swap floating into fixed.

A more detailed explanation goes as follows. When a trading desk sells an option, it places the premium in an account called the 'hedging account'. This account earns interest from the firm's central funding desk and the interest earned will typically be close to the firm's overall funding rate. To hedge the risk arising from changes in this outer discount factor, the desk should make an internal or external IR swap into a fixed rate with maturity at the option expiry date.

It can then be shown that this swap guarantees that, with no other change taking place in the market, the hedge account will earn the fixed rate r in the outer discount factor e^{-rt}. The demonstration goes as follows. Let

$$(31) \qquad P = FV \times e^{-rT}$$

where P is the option premium paid, r here is the long term rate earned on the option account, and FV is the future value of the option given by the *un*discounted Black formula.

Now suppose the long-term interest rate r changes, but there is no change in the forward price F. Such a circumstance would occur where the spot rate S changed by an amount ΔS in such a way that F remained constant under the formula connecting S with F, i.e.

$$(32) \qquad \Delta S = S(e^{-\Delta rT} - 1)$$

where Δr is the change in discount rate. In this situation F will remain the same and hence the future value FV of the put option will also remain the same. The change ΔP in the value of the option premium will then be a simple discount function:

$$(33) \qquad P + \Delta P = FV[e^{-(r+\Delta r)T} - e^{-rT}].$$

Assuming the amount P is currently held in the hedging account, we could replicate the change ΔP if P were invested in a long dated zero-coupon bond with maturity T. In practice the same effect can be achieved by investing P at the firm's short-term funding rate, but swapping the short-term floating payments into a zero-coupon swap.

Projection rate risk

In the previous example we assumed that the forward price remains constant while the spot price changes, i.e., a rise in long term interest rates will force the spot price lower, while a fall in the interest rate forces the spot price higher. This effect might be explained by a market expectation of unchanged future nominal rental cashflows, whose discounted present value would fall or rise according to the long-term interest rate operating as a discount factor.

The opposite case can also occur, i.e., we could have a situation where the spot remains steady, but the forward changes due to the way in which the interest rate operates as a projection factor.

Using the standard formula for the forward house price (i.e., (30)), and assuming constant q, the return on the forward is calculated as follows.

(34) forward return $\approx \Delta HP + (\Delta IR - \Delta q) \times T$

where $\Delta HP = \ln((S + \Delta S)/S)$ and T is the maturity of the forward at any point in the historical time series for the given combination of interest rate (IR), deferment rate (q) and house price index (HPI).

A proof of (34) is provided in Appendix 1.

Three key points follow from (34).

Four risk factors

The first is that we now have four risk factors (Index, achievement rate, interest rate and deferment rate) impacting the forward price.

Correlations

The second is that we need to consider their correlations. We are now interested in particularly (a) the correlation between the house price index and the interest rate, and (b) the correlation between the house price index and the deferment rate.

Let's consider each of these in turn.

(a) Correlation between house price index and interest rate

We have already assumed that the correlation between the Index and the achievement rate is zero. The table below shows the correlation between the 10 year interest rate (which we take as a benchmark for the whole term structure) and the housing index, for 10 representative countries.

Table 2: Correlation between 10Y Interest Rate and Index

Country	$\rho_{IR,INDEX}$
AUS 10Y	0.22
CAN 10Y	-0.05
GER 10Y	0.11
ESP 10Y	0.06
FRA 10Y	0.16
GB 10Y	**0.10**
IRL 10Y	-0.03
SWE 10Y	0.14
US 10Y	0.03
JP 10Y	0.27

Source: OECD (10Y interest rate) and Dallas Fed (House Price indices)

The takeaway points from this table are that the correlations between interest rates and house price indices are generally low and that a reasonable correlation for the UK would be zero.[18]

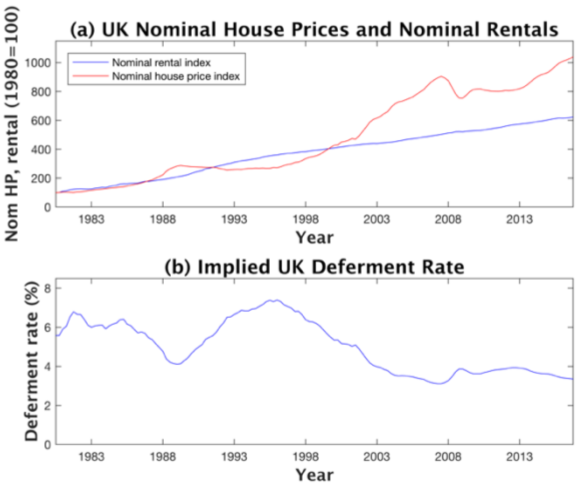
(b) Correlation between house price index and q

Equation (34) indicates that the deferment rate q is a further source of volatility. Take equation (11), which says that the deferment rate is equal to the rental yield divided by the house price, then replace the house price by the HP index and add a time t subscript. We then obtain:

(35) $$q_t = d_t / HP_t$$

where d_t is the aggregate nominal rental. We have no time series data on aggregate nominal rentals, but we can estimate their change using rental and house price indices. Data from OECD suggest that the deferment rate q is not constant (the annual volatility of q for the UK is of the order of 0.3%) and that changes in q are *negatively* correlated with changes in the index. These effects are shown in Figure 3:

Figure 3: UK Nominal House and Rental Indices and Implied Deferment Rate



Sources: OECD

As one of us comments in a blog posting in early 2019:[19]

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[18] The low correlations reported in Table 2 are a bit of a surprise, considering that a lower interest rate immediately transforms into higher affordability and therefore - in the absence of new supply – into higher house prices. One reason could be that Table 2 looks at 10yr rates whereas the key driver might be short-term rates. Were the correlations between interest rates and house prices higher, the resulting 'combined' volatilities (of which more below) would be higher as well.

[19] Buckner (2019).

> When I worked at the PRA on the paper that became CP 13/18, I had assumed that the deferment rate stays roughly constant. The rationale is that if rentals are expected to increase, this would increase the market value of properties, *all other things being equal*. But all other things aren't equal: there is strong evidence that *nominal* rentals track price inflation, and also strong evidence that interest rates anticipate inflation. So an increase in expected nominal rentals should correlate strongly with an increase in the interest rate used to discount the same rentals, and the rental yield, hence the deferment rate, should remain roughly constant.[20] I assumed this, and I imagine the PRA assumed this too.

A possible explanation for the volatile q rate and the negative correlation with house prices might then go as follows. Nominal house prices, which in theory should reflect the net present value of all future nominal (net) rental cashflows, tend quickly to anticipate – perhaps to over-anticipate – future upward or downward changes in rentals. Nominal rentals are sticky however and respond slowly.[21] Thus the large fall in house prices which occurred in the housing recession of the early 1990s was not reflected in rental prices, which continued to rise slowly, and so q rose in that period. Conversely, the significant rise in house prices from the late 1990s to 2007 was notably higher than the rise in rentals, so q fell over this later period.

As an aside, this combination of a volatile q that is negatively correlated with house prices has an interesting policy implication. If house prices go up, the loan-to-value of an existing equity release mortgage will fall, which will decrease the cost of the NNEG. At the same time, the graph above suggests the deferment rate will also fall, which will make the NNEG even cheaper, given that the deferment rate is the main driver of NNEG cost. Conversely, a fall in house prices will make the NNEG more expensive because of the fall itself, and will then make the NNEG even more expensive because of the implied rise in the q rate. The cost of the embedded guarantee is thus doubly geared to the state of the housing market. The PRA would appear to be still unaware of this double exposure, which has implications for how it should design its capital requirement regime for equity release. But we digress.

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[20] Using the Dividend Discount Model, the deferment rate q equals the net rental yield d/S (see (18) above). We show elsewhere (see Buckner and Dowd (2020a, p. 107)) that q also equals $r + \pi + \lambda - g$. Using the well known Fisher Equation, we can decompose the nominal interest rate r into the sum of $r^{real} + inf$, where r^{real} is the 'real' rate of interest and inf is the expected inflation rate. However, we can apply a similar reasoning to g as well, giving us $g = g^{real} + inf$, where g^{real} is the growth rate of 'real' or inflation-protected dividends. We then obtain $q = r^{real} + inf + \pi + \lambda - (g^{real} + inf) = r^{real} + \pi + \lambda - g^{real}$ from which we see that the effects of expected inflation cancel out.

[21] This effect is well known in the literature, although there is no consensus on the explanation. For example, Campbell and Hercowitz (2009) find that "movements in U.S. house price-rent ratios cannot be fully explained by movements in subsequent rent growth" For a review of the literature, see Gelain and Lansing (2013).

Nor is this negative correlation effect unique to the UK. Table 3 shows evidence for a strong and consistent negative correlation between the deferment rate and the house price index of our ten countries:

Table 3: Correlation between q and Index

Country	$\boldsymbol{\rho_{q,INDEX}}$
AUS	-0.90
CAN	-0.95
GER	-0.79
ESP	-0.88
FRA	-0.84
GB	**-0.82**
IRL	-0.43
SWE	-0.80
US	-0.86
JP	-0.92

Source: OECD (10Y interest rate) and Dallas Fed (House Price indices)

Volatility term structure

The third important corollary of equation (34) is that it implies that there is a volatility term structure. For example, other things being equal, the effect of a 10 bp change on a contract with 3 months to maturity will be 10 x 3/12 = 2.5bp, which is trivial. But the impact of the same bp change on a 30 year forward will be 10 x 30 = 300bp, which is a whole lot more. This changing sensitivity throughout the life of the contract means that the impact on volatility caused by changes in the interest rate is not constant, and the same applied to changes in the deferment rate. Instead, mapped against time, the volatility starts high when the maturity is far from maturity and falls towards zero as the contract approaches expiry. Mapped against maturity, the volatility starts from zero and increases as the maturity gets larger.

We now wish to determine the average lifetime volatility of the contract. If X is the series of maturities and Y is the series of forward returns, it can then be shown that the volatility of the product $\sigma(XY)$ is the following simple function:

(36) $$\sigma(XY) = \sigma(Y) \times T/\sqrt{3}$$

where $\sigma(XY)$ is the volatility of a time series of returns of the forward contract, and $\sigma(Y)$ the volatility of the interest rate. A proof is given in Appendix 2. Then the volatility of the returns on the forward contract is directly proportional to T.

Total Forward Volatility

Finally, we might consider the effect of all four risk factors (Index, interest rate, q and achievement rate) in the forward rate (34) to give what we might call the total forward volatility. We can do so by estimating a correlation matrix between the four risk factors as shown in Table 4:

Table 4: Correlation Matrix for the Four Main Risk Factors

	Index	IR	q	AR
Index	1.00	0.00	-0.82	0.00
IR	0.00	1.00	0.00	0.00
q	-0.82	0.00	1.00	0.00
AR	0.00	0.00	0.00	1.00

Table 5 shows the volatilities for the component risk factors:

Table 5: Volatilities of Component Risk Factors

Component Volatility	Value
σ_{INDEX}	13%
σ_{AR}	8.5%
σ_{IR}	0.58%
σ_{q}	0.17%

We next combine the correlations in Table 4 with the component volatilities in Table 5 and then apply (36) to obtain the term structure for the total forward volatility shown in Table 6:

Table 6: Term Structure of Total Forward Volatility

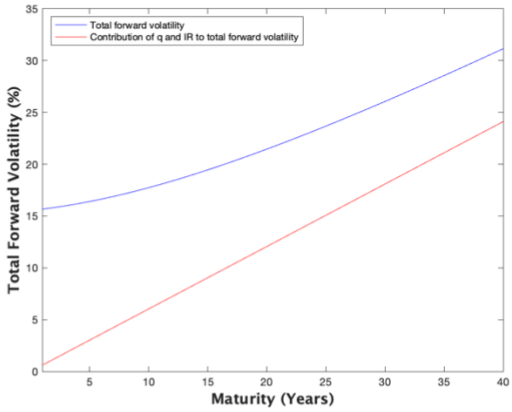
T	Total Forward Volatility
1	15.66%
5	16.39%
10	17.73%
15	19.45%
20	21.45%
25	23.67%
30	26.05%

If we worked with these results, we would apply a 15.66% volatility to the put for decrement $t = 1$, a 16.39% volatility to the put for decrement 5, and so on, and a 26.05% volatility to the put for decrement 30.[22]

Figure 4 shows a plot of the total forward volatility over a horizon of up to 40 years.

[22] A more comprehensive approach would give a more detailed consideration of the correlation between the index and the achievement rate.

Figure 4: Term Structure of Total Forward Volatility



The blue plot shows total forward vol, which starts at a little over 15% for a maturity of $T = 1$ and rises to a little over 31% for $T = 40$. The red line shows the contribution to that total forward vol made by the q and interest rate risk factors. These contributions start (for $T = 1$) at close to zero, but rise linearly (see equation (9.14)) with T, reaching over 24% for $T = 40$. Thus, as T increases, the q and interest rate risk factors become increasingly important drivers of total forward volatility.

Obtaining a Single Volatility Estimate for Use in All Put Decrements

Having established that, in principle, each put decrement requires its own (potentially) different vol estimate, it is still possible to use a single volatility for all puts, *provided* one uses a single volatility calibration that is appropriate to the borrower's age and gender. To obtain such a calibration, one could use the expected volatility obtained by weighting each volatility by its exit prob or decrement probability. The formula for the expected volatility σ^e is then

(37) $$\sigma^e = \sum_t [exit\ prob_t \times \sigma_t].$$

Table 7 shows these expected volatilities against borrower age: for males 26.1% for age 55, 27.3% for age 70 (and hence our earlier baseline single volatility recommendation for males aged 70) and so on. These expected volatilities give results that are very close to the results that one would have obtained had one used the full volatility term structure.

Table 7: Expected Volatilities for Different Ages

Borrower Age	Expected Volatility (Males)	Expected Volatility (Females)
55	26.1%	27.3%
60	23.8%	24.9%
65	21.7%	22.7%
70	20.0%	20.8%
75	18.6%	19.2%

80	17.6%	18.0%
85	16.9%	17.1%
90	16.4%	16.5%

Notes: Exit probabilities as per Figure 1.

5. Illustrative Valuations

We now build an ERM and NNEG valuation model based on plausible input parameter calibration values.

The baseline parameter inputs are:

- Current age of customer = 70, a typical age for ERMs.[23]
- Loan to value ratio = 40%.[24]
- Risk-free rate r = 0.25%.
- ERM loan rate $l = 4\%$.[25]
- Deferment rate $q = 4.2\%$.
- Volatility $\sigma = 20\%$ for males aged 70.

All rates are in % p.a.

We assume an illustrative house price of £100 which, combined with the assumed loan to value ratio of 40%, implies a loan amount = £40.

The death/exit probabilities are derived from projections of future mortality rates obtained using the M5 version of the Cairns-Blake-Dowd mortality model (see Cairns *et alia*, 2006, 2009) calibrated on England & Wales male mortality data for the period 1971 to 2017 and spanning ages 55 to 89. The data are taken from the Life and Longevity Markets Association database. The M5-CBD model is particularly suitable for old age projections and its goodness of fit and performance evaluation are assessed in Cairns *et alia* (2011) and Dowd *et alia* (2010a,b).

Our baseline NNEG valuation results are shown in Table 8:

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[23] Implicitly, we are assuming a single male just turned 70. In the case of a single female, we would expect death/exit to occur somewhat later, which would increase the value of the NNEG. In the case of a couple, we would expect even later exit, when the longest surviving member of the couple exits the house.
[24] A 40% LTV ratio for a 70-year old appears to be approximately in line with current industry practice for new ERM loans. A more detailed analysis of the LTV ratio is given in Buckner and Dowd (2020b, chapter 4).
[25] The Equity Release Council report that the average loan rate fell to 4.01% during 2020Q4 (Equity Release Council, 2021). This point made, loan rates are trending downwards and there is considerable variation.

Table 8: Baseline ERM/NNEG Valuations

Current House Price	*Loan Amount*	L	$NNEG$	ERM
£100	£40	£74.76	£35.08	£39.68

Notes: L is the present value of the loan component of the Equity Release Mortgage.

Given the age of the customer, the expected present value L of the perfectly collateralised loan is £74.76. $NNEG$ is valued at £35.08 and so the value of the ERM, ERM, is equal to £74.76 – £35.08 = £39.68.

Sensitivities of Valuations to Input Parameters

Table 9 shows the sensitivities of L, $NNEG$ and ERM to changes in key parameter inputs. These are expressed in elasticity form, i.e., where the elasticity of the relevant output with respect to a change in an input is the % change in the output divided by the % change in the input.

Table 9: Sensitivities of Valuations in Elasticity Form

Elasticity wrt	L	$NNEG$	ERM
r	-0.04	-0.08	-0.01
l	0.72	1.30	0.20
q	0	0.35	-0.43
σ	0	0.23	-0.28
LTV	1	1.48	0.41

Notes: As per Table 8.

These results indicate that NNEG valuations are highly sensitive to changes in the l and LTV input parameter calibrations. It is also interesting to note that the ERM valuations are much less so, because of the offsetting impacts on the loan value and NNEG valuations.

Table 10 gives the same valuations for age 70 as percentages of the initial loan amount, £40:

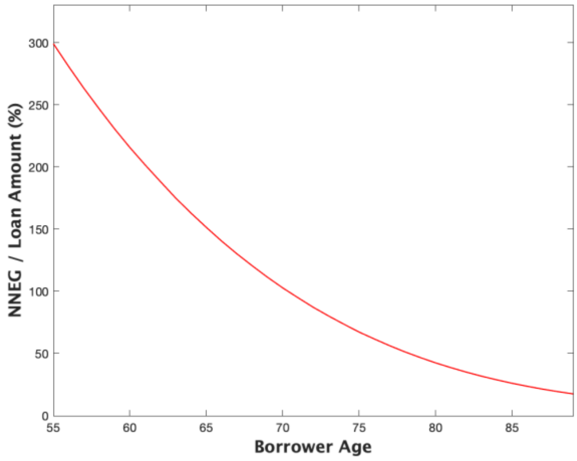
Table 10: ERM and NNEG Valuations as Percentages of Loan Amount: Male Age 70

Valuation Approach	*NNEG*	*ERM*
Black '76 (using expected vol)	87.7	99.2

Notes: Based on baseline case calibrations.

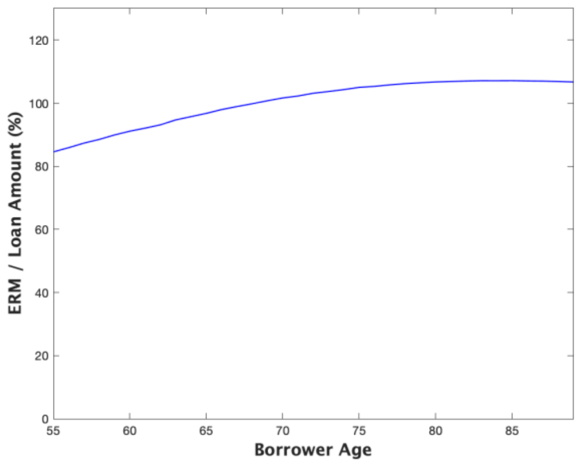
Figures 5 and 6 give plots of NNEG/loan values against borrower age and ERM/loan values against borrower age.

Figure 5: NNEG/Loan Ratios Vs Borrower Age



Notes: As per Table 10.

Figure 6: ERM/Loan Ratios Vs Borrower Age



Notes: Asper Table 8.10

One notices that the ratios of $NNEG$ to loan amount fall sharply with age. However, what is most significant are the low ratios of ERM to loan amount, which suggest that ERMs generate low (and in some cases, negative) profits to lenders, especially loans to younger borrowers.

6. Conclusions

This paper proposes a new approach to the valuation of NNEGs and to the valuation of ERMs that include such guarantees. The proposed approach is in the market consistent tradition and is based on a combination of the Black '76 put option model and the M5-CBD mortality model. The paper makes two distinct contributions to the existing literature. First, it provides a proof that the deferment rate is equal to the net rental yield, and also suggests that the latter is quite straightforward to calibrate. Second, it proposes a new approach to the estimation of the volatility inputs to the nneglet put decrements. The proposed approach produces a volatility term structure and indicates that the volatilities inputted to those nneglet put decrements depend on the age and gender of the borrower. It provides some illustrative valuations that suggest that ERMs are less profitable than is commonly supposed. These results have interesting ramifications for actuarial practice in and prudential regulation of the UK equity release sector.

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Appendix One: Proof of Approximation (34)

The forward rate F at any time t and for any maturity T is as follows:

(A1.1) $$F_{t,T} = S_t e^{(r(t,T)-q(t,T))T}$$

where S_t is the spot price at time t, $r(t,T)$ is the interest rate of maturity T at time t and $q(t,T)$ is the deferment rate of maturity T at time t. With the passage of time Δt, the forward rate will change as a result of changes in S, r and q, and of course with the passage of time itself. Thus

(A1.2) $$F_{t+\Delta t,T-\Delta t} = S_{t+\Delta t} e^{\left(r(t+\Delta t,T-\Delta t)-q(t+\Delta t,T-\Delta t)\right)(T-\Delta t)}$$

This expression is fairly complex, but we can make a number of simplifying assumptions as follows. First, we can assume that the term structure of both r and q is continuous. We have assumed throughout a flat term structure q, so it follows that $q(t+\Delta t, T-\Delta t)$ equals $q(t+\Delta t, T)$. We cannot assume that the term structure of interest rates is flat, because it will usually slope upwards or downwards at any time. However, we can reasonably assume that changes in the term structure will make no significant contribution to volatility. That is, a change over 1 month to the 10 year interest rate will not be significantly different from the change in the 9 year 11 month interest rate. Thus $r(t+\Delta t, T-\Delta t)$ will be approximately equal to $r(t+\Delta t, T)$, for small Δt. Hence

(A1.3) $$F_{t+\Delta t,T-\Delta t} \approx S_{t+\Delta t} e^{\left(r(t+\Delta t,T)-q(t+\Delta t,T)\right)(T-\Delta t)}$$

The outer term $(T-\Delta t)$ can also be eliminated, as it represents a constant carry through time. As time passes, if r is greater than q, the forward price will gradually fall, or if r is less than q, the forward price will gradually rise. But volatility corresponds to the mean *difference* from the average, whereas the carry term will be close to the average itself. Hence

(A1.4) $$F_{t+\Delta t,T-\Delta t} \approx S_{t+\Delta t} e^{\left(r(t+\Delta t,T)-q(t+\Delta t,T)\right)T}.$$

We assume that the determinants of forward volatility are the changes in spot, interest rate and deferment rates alone, and that the passage of time is an insignificant contribution to volatility.

To determine the volatility, we must first determine the forward price return:

(A1.5) Forward price return = ln $(F_{t+\Delta t,T-\Delta t}/F_{t,T})$.

Substituting from the equation above:

(A1.6) $\ln(F_{t+\Delta t,T-\Delta t}/F_{t,T}) =$

$$\ln\left[S_{t+\Delta t}\, e^{\left(r(t+\Delta t,T)-q(t+\Delta t,T)\right)T}\right] - ln[S_t e^{\left(r(t,T)-q(t,T)\right)T}] =$$

$$ln[S_{t+\Delta t}/S_t] + [r(t+\Delta t,T) - r(t,T) + q(t,T) - q(t+\Delta t,T)] \times T.$$

Now make the simplifying assumptions that $r(t+\Delta t,T) - r(t,T) = \Delta r_t$ and $q(t+\Delta t,T) - q(t,T) = \Delta q_t$. We then obtain:

(A1.7) forward return ≈ $\varDelta HP_t + (\varDelta r_t - \varDelta q_t) \times T$

which was to be proved, where $\varDelta HP = \ln((S + \Delta S)/S)$.

Appendix Two: Proof of Equation (37)

We need to determine the volatility of a time series of prices for a forward contract, given that the maturity T of the contract is constantly decreasing. Assume the following standard result for two independent variables X and Y:[26]

(A2.1) $$Var(XY) = Var(X)Var(Y) + Var(Y)E[X]^2 + Var(X)E[Y]^2$$

Let X be the series of maturities, and Y be the changes in interest rate Δr_t (or deferment rate Δq_t). Assume that the average interest rate or deferment rate change is zero, i.e. that $E[Y] = 0$.

(A2.2) $$\begin{aligned} Var(XY) &= Var(X)Var(Y) + Var(Y)E[X]^2 + Var(X)E[Y]^2 \\ &= Var(Y)[Var(X) + E[X]^2]. \end{aligned}$$

Then we can treat the series of maturities as a uniform distribution from the starting maturity T down to zero. The variance $Var(X)$ and the average $E[X]$ of a uniform distribution over the interval (x, y) are as follows:

(A2.3) $$Var(X) = (y - x)^2/12 = T^2/12$$

(A2.4) $$E[X] = T/2.$$

Substituting:

(A2.5) $$\begin{aligned} Var(XY) &= Var(Y)[Var(X) + E[X]^2] \\ &= Var(Y)[T^2/12 + T^2/4] \\ &= Var(Y) \times T^2/3 \end{aligned}$$

(A2.6) $$\sigma(XY) = \sqrt{Var(XY)} = \sigma(Y) \times T/\sqrt{3}$$

which was to be proven.

[26] This result is proven in Goodman (December 1960, 708).