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Option Pricing in the Presence of a Reflecting Barrier

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Abstract

A number of recent papers examine option pricing in the presence of a bound on the price of the underlying. One of these is Thomas (2020) which examines the valuation of the no-negative equity guarantees in equity release mortgages based on the premise that policymakers set a floor or reflecting barrier to future house prices. This approach is flawed, however. First, it can give valuations that are indefensible because they violate the rational valuation principles for equity release set out by the United Kingdom Prudential Regulation Authority. Second, the introduction or removal of the reflecting barrier policy is likely to change both the current value of the underlying price and the value of the volatility, and Thomas does not allow for such impacts. It is also possible to show that Thomas' key valuation equation does not hold in the case of a deep in the money put option, which implies that the option pricing equation itself is incorrect. Finally, the Thomas approach violates the no-arbitrage principle. Our results call into question the validity of the reflecting barrier approach not just in the equity release context, but in other contexts too.

Keywords: Black '76 model; option pricing; reflecting barrier; equity release; no negative equity guarantee.

JEL Classification: G2, G3

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1. Introduction

A variety of recent models seek to price options in the presence of some barrier on the value of the underlying. We can think of such barriers as reflecting ones: as the underlying price approaches and then touches the barrier, the price is 'reflected' back so that the barrier is never breached. The seminal example of this approach is Veestraeten (2008) on the valuation of options when the underlying is subject to a lower boundary. Other examples are Hertrich and Zimmermann (2017) who examine such barriers in an FX context and Thomas (2020) on the valuation of NNEG guarantees in an ER context. The message coming from these studies is that such barriers are intuitive and plausible, and have quantitatively significant impacts and potentially important policy implications.

This article examines the correctness of reflecting barrier approaches to option pricing taking the Thomas approach as an example. To elaborate: Thomas considers the effect of a hypothetical reflecting barrier policy on the valuation of the no-negative-equity guarantees (NNEGs) in equity release mortgages (ERMs). An ERM is a loan made to an older property-owning borrower that is mortgaged by their property and repaid when they permanently leave it. A NNEG is a guarantee made by the lender to the borrower to cap the maximum amount owed at the value of the property when the borrower leaves it. The policy he considers is a presumed commitment by policymakers to support future house prices and is modelled in terms of a reflecting barrier expressed as a fraction of the current level of house prices. If the reflecting barrier is X% of current prices, the presumption is that policymakers will ensure that future house prices will not fall below X% of their current value. Applying this approach to NNEG valuation, his key point (p. 2) is that 'assuming a reflecting barrier even as much as 50% below today's level can substantially reduce the value of [the] NNEG.'

The proposed approach involves valuation of a bull put spread, a well-known option trade. This trade involves selling a put with a high strike price and buying a put with a lower strike price. The presence of the latter option can lead to a NNEG valuation that is considerably lower than the valuation one would obtain under the conventional Black '76 approach. Thus, Mr. Thomas provides an important challenge to the Black '76 approach to NNEG valuation.

In this article we do not address the issue of whether policymakers have adopted or are able to successfully implement Thomas's assumed policy of supporting the housing market. Instead, we argue that his NNEG valuation approach is unsound in principle, even if one accepts that the policy exists and that everyone understands how it would be implemented. The next section outlines his proposed approach, and the following sections discuss specific problems with it, namely: (a) that it can give indefensible valuations; (b) that the introduction of a reflecting barrier policy is likely to change the values of both the underlying price and the volatility, yet Thomas's key equation assumes that both values remain the same, regardless of whether there is a barrier or not. It is also possible to show that (c) Thomas' key equation, reproduced as equation (2) below, does not hold in the case of a deep in the money put option, which implies that the equation itself must be incorrect; and (d) his proposed valuation approach violates the no-arbitrage principle. The final section discusses the implications of our analysis for both the Thomas approach and for reflecting barrier cases in general.

2. Analytical Framework

Assume that we are dealing with an ERM loan that will be repaid in t years. Let L_t be the present value of a risk-free loan that matures in t years, and let put_t be a put option with current underlying price S and strike price X equal to the rolled-up loan amount, which will expire in t years. Then ERM_t, the present value of the ERM loan, is given by

(1) $$ERM_t = L_t - put_t$$

where put_t is the value of the NNEG for year t.

We are then dealing with two alternative valuations of put_t: the B76 valuation, $B76\, put_t(X, 0)$, which is the value, in a *non-barrier world*, of a European put struck at X; and the Thomas (T20) valuation, $T20\, put_t(X, b)$, which is the value, in a *barrier world*, of the *same* put.

Thomas assumes that the policymaker is able to enforce a (fixed) reflecting barrier b on the underlying and implicitly assumes that this 'reflecting barrier' policy and the value of b are known. He further assumes that $0 < b < \min\,(S, X)$.

Thomas then derives the following equation (see his (3), but using our notation) for his valuation:

(2) $$T20\, put_t(X, b) = B76\, put_t(X, 0) - B76\, put_t(b, 0) + Adjustment$$

where: $T20\, put_t(X, b)$ is the value of a put struck at X in a world in which there is a reflecting barrier b; $B76\, put_t(X, 0)$ is the B76 value of the same put struck at X in a world in which there is no reflecting barrier or, equivalently, in a world in which the reflecting barrier is 0; $B76\, put_t(b, 0)$ is the B76 value of a put struck at b with reflecting barrier 0; and $Adjustment$ is a (complicated) term that we do not reproduce here for space reasons. Note that the impact of the latter two terms is to reduce $T20\, put_t(X, b)$ relative to $B76\, put_t(X, 0)$. Note, too, that in the special case where $b = 0$, the values of these latter two terms become zero and $T20\, put_t(X, b)$ is equal to $B76\, put_t(X, 0)$.

So we have our ERM_t valuation

(3) $$BD20\, ERM_t = L_t - B76\, put_t(X, 0)$$

and the Thomas ERM_t valuation

(4) $$\begin{aligned} T20\, ERM_t &= L_t - T20\, put_t(X, b) \\ &= L_t - B76\, put_t(X, 0) + B76\, put_t(b, 0) - Adjustment. \end{aligned}$$

3. Thomas Model Violates PRA Equity Release Valuation Principle II

The first problem with the Thomas valuation model is that for some plausible combinations of input values, his model violates one of the rational valuation principles set out by the UK Prudential Regulation Authority (PRA). To explain, in its *Supervisory Statement* SS 3/17 published in 2017, the PRA set out certain principles relating to the valuation of ERM portfolios. These principles establish model-free bounds on any proposed ERM valuations. By 'model-free bounds,' we mean bounds that do not depend on any choice of option-pricing model. These bounds can then be used to test any proposed ERM valuation model, the point being that if the model produces valuations that violate these bounds, then the model is unreliable.

We are interested here in Principle II, which states:

> [Principle II] The economic value of ERM cash flows *cannot* be greater than either the value of an equivalent loan without an NNEG or the present value of deferred possession of the property providing collateral … (our emphasis)

Put differently, Principle II states that ERM_t is bounded above by min $[L_t, PV(F_t)]$, where $PV(F_t)$ is the present value of a period t forward contract on the mortgaged property. We are particularly interested in the second part of Principle II, i.e.,

(5) $$ERM_t \leq PV(F_t) \text{ for all } t.$$

The validity of Principle II is shown in Buckner and Dowd (2020, ch. 20).

To illustrate further, assume the following plausible inputs shown in Table 1.

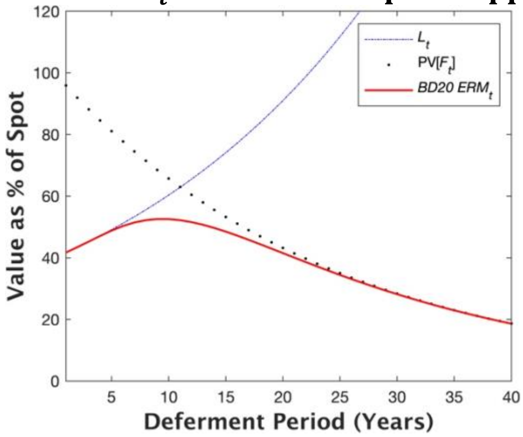
Table 1: Hypothetical Inputs for ERM_t

Initial house price	S	1
Loan to value ratio	LTV	40%
Barrier value	b	52%
Risk-free interest rate	r	0%
Deferment rate	q	4.2%
Loan rate	l	4.11%
Volatility	v	13%

Notes to Table 1: (a) Loan to Value ratio: Chosen value is in line with the 'age minus 30' rule of thumb which approximates industry practice and applied to a new borrower aged 70. (b) Risk free interest rate: Bank Rate is 0.1% at time of writing. (c) Deferment rate: Best estimate recommended by Buckner and Dowd (2020, p. 37). (d) Loan rate: average loan rate reported by the Equity Release Council (2020). (e) PRA 'central estimate' (see CP 13/18, p. 9).

The following chart gives a graphic illustration of how our ERM_t series compares to the Principle II bounds, based on the calibrations in Table 1.

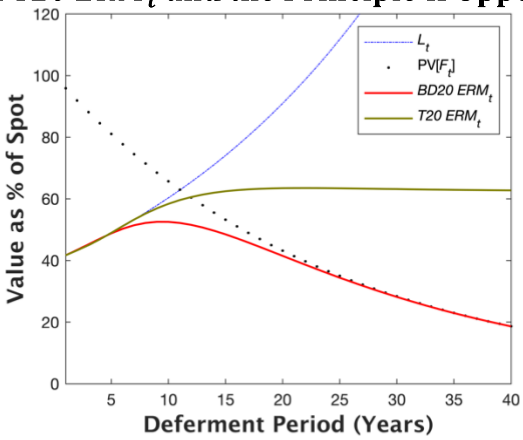
Figure 1: *BD20* ERM_t and the Principle II Upper Bound



Principle II requires that the ERM_t line should be bounded above by the other two lines, and we see that the *BD20* ERM_t line shown in red does in fact satisfy this requirement.

Figure 2 adds the corresponding plot of the $T20\ ERM_t$ line in olive green.

Figure 2: *T20* ERM_t and the Principle II Upper Bound



We see that from $t = 12$ onwards, the $T20\ ERM_t$ line lies above the $PV[F_t]$ and so violates Principle II.

To drill down further, consider the case where $t = 25$. We now go through the following calculations:

(6) $$L_{25} = S.LTV.e^{(l-r)t} = 100\% \times 40\% \times e^{(4.11\%-0\%)\times 25} = 112\%$$
(7) $$K_{25} = S.LTV.e^{lt} = 100\% \times 40\% \times e^{4.11\% \times 25} = 112\%$$

where $X = K_{25}$ refers to the strike price at $t = 25$;

(8) $$F_{25} \quad = Se^{(r-q)t} = e^{(0\%-4.2\%)\times 25} = 35\%$$
(9) $$PV[F_{25}] = Se^{(-0\%)t} \times 35\% = 35\%.$$

Then consider the BD '20 valuations:

$$
\begin{aligned}
&(10) \qquad B76\, put_{25}(X,0) &&= 77\% \\
&(11) \qquad BD20\, ERM_{25} &&= L_{25} - B76\, put_{25}(X,0) \\
& &&= 112\% - 77\% \\
& &&= 35\%.
\end{aligned}
$$

So, for $t = 25$, $BD20\ ERM_t = 35\% = PV[F_t]$ and passes the Principle II test that $ERM_t \leq PV(F_t)$.

Now consider the Thomas '20 valuations:

$$(12) \qquad B76\, put_{25}(b,0) = 21\%$$

where b = 52% from Table 1

$$
\begin{aligned}
&(13) \qquad Adjustment &&= -8\% \\
&(14) \qquad T20\, put_{25}(X,b) &&= B76\, put_{25}(X,0) - B76\, put_{25}(b,0) + Adjustment \\
& &&= 77\% - 21\% - 8\% \\
& &&= 48\% \\
&(15) \qquad T20\, ERM_{30} &&= L_{25} - T20\, put_{25}(X,b) \\
& &&= 112\% - 48\% \\
& &&= 64\%.
\end{aligned}
$$

Thus, $T20\ ERM_{25} = 64\% > PV(F_{25}) = 35\%$ and so violates Principle II: the ERM based on the Thomas valuation approach gives indefensible valuations.

4. Hypothetical Valuations Under Alternative States of the World

A second problem is that Thomas's key equation requires assumptions about how the same option would be valued under alternative states of the word, only one of which can be observed. The key equation is (2), which is reproduced below

$$(2) \qquad T20\, put_t(X,b) = B76\, put_t(X,0) - B76\, put_t(b,0) + Adjustment.$$

We have interpreted this equation as follows. The price of an ordinary European put option struck at X in a world *with* a barrier at b is equal to the B76 price of the same option in a world where there is no barrier, less the B76 price of the same put option struck at b, also in a world with no barrier, plus an adjustment.

Assume we are in a world in which there is a barrier. If so, how would we know what the price of the same option would be if the barrier did not exist? Or conversely, if we are in a world without a barrier, how would we know the price of the option if a barrier were introduced?

Part of the problem is that the removal or introduction of a barrier is likely to affect the spot price. We see this effect in interventions in the currency market. The successful imposition of a lower barrier is likely to strengthen the exchange rate and its removal is likely to weaken the exchange rate, as we saw in the case of Sterling in September 1992 and the Russian Rouble in August 1998. By putting a floor under future prices, the introduction of a reflecting barrier will be likely to raise the current underlying price (e.g., from £1 to £1.2, say) but Thomas fails to take account of any such impact. Thus, the comparison of T20 and conventional B76 values based on the same (current) value of the underlying is invalid.

Successful barrier-type interventions also tend to reduce the market volatility, as seen for example in the case of the European Exchange Rate mechanism prior to the introduction of the Euro. It is therefore invalid to calibrate the T20 model with a single volatility and then compare the output from that model so calibrated to that from a B76 pricing model based on the same volatility calibration. Put another way, the introduction or removal of the hypothetical reflecting barrier itself changes the behaviour of the volatility and Thomas's calibration fails to take this impact into account either.

The point is that, given that we are in one world, we have little or no knowledge of what calibrations under the alternative world would be like.

5. Deep in the Money Puts

However, it also turns out that there is a special case in which the precise calibrations of S and the volatility do not matter, and in that case, it turns out that the Thomas approach gives the wrong answer. This special case is where the put is deep in the money, where the option value is almost entirely unaffected by volatility. The value of such an option is almost entirely dependent on its intrinsic value, and almost not at all on its time value, which reflects the probability (in this case, close to zero) of the asset price crossing the strike. But the factors which determine the intrinsic value are the same, whether or not the barrier exists. Hence, given $0 < b < S \ll X$, then

(16) $$T20\, put_t(X, b) \approx X - F$$

(17) $$B76\, put_t(X, 0) \approx X - F$$

which implies

(18) $$T20\, put_t(X, b) \approx B76\, put_t(X, 0).$$

Given that by assumption the volatility of the market is greater than zero, then $B76\, put_t(b, 0)$ is greater than zero. Given also that $Adjustment < 0$, then (2) implies

(19) $$T20\, put_t(X, b) < B76\, put_t(X, 0)$$

However, (18) and (19) cannot both be correct. And since (18) *must* hold in the deep in the money case, then clearly (2) and its corollary (19) cannot not. But if Thomas's equation (2) were correct, it would hold in all cases, including this case. Therefore, equations (2) and (19) must be incorrect.

To give a numerical illustration, suppose we set $S = 52.1\%$, which is just over the barrier b, whose value is 52%. Also set $q = 0.1\%$ and $\sigma = 0.1\%$. These are all logically permissible calibrations. For $t = 25$, we can show, using the same approach as in section 3, that $F_{25} = 50.8\%$ and $X = K_{25} = 58.2\%$. Therefore, applying (17), $B76\, put_t(X, 0) \approx 58.2\% - 50.8\% = 7.4\%$. The correctness of this valuation can be confirmed by applying the B76 put formula directly. Applying (17) or (18), we then obtain $T20\, put_t(X, b) \approx 7.4\%$. But if we apply the Thomas formula for $T20\, put_t(X, b)$, then we obtain the result that $T20\, put_t(X, b) = 6.2\%$. Since we know that the correct value of $T20\, put_t(X, b)$ is 7.4%, then we can infer that the Thomas formula gives an incorrect result for this set of parameter calibrations. From which we can infer that the Thomas formula is wrong, period.

Now recall our earlier finding that the Thomas valuation can violate the Principle II bounds. These violations occur precisely where the put options are deep in the money. We can now explain why the Thomas valuations violate the Principle II bound: they do so because they mis-specify the intrinsic value of the option when it is deep in the money. We see this misspecification in the cases of the long maturity puts in Figure 1.

6. Violation of the No Arbitrage Principle

There is a final problem as a consequence: the model appears to violate the no-arbitrage principle.

Suppose that these put options are traded instruments. In that case, consider our earlier numerical example (see pp. 4-5 above). Based on $t = 25$, $b = 52\%$ and $X = 112\%$, we obtain the two option valuations $B76\, put_{25}(X, 0) = 77\%$ (see eqn (10)) and $T20\, put_{25}(X, b) = 48\%$ (see eqn (14)). We sell the put at the former valuation and buy the put at the latter valuation. Since we have bought and sold the same option at different prices, we make a guaranteed risk-free profit equal to the difference between the two option prices, which is 77% minus 48% = 29%.[1]

A potential objection to this argument is that no-one would trade at the higher $B76\, put_{25}(X, 0)$ valuation. In that case, we would still buy the put at the lower $T20\, put_{25}(X, b)$ valuation, but instead of selling a put at the $B76\, put_{25}(X, 0)$ valuation, we would construct a synthetic put position using a trading strategy based on the B76 delta for a short put struck at X. Assuming that the synthetic position gives a perfect replication of an actual put based on the $B76\, put_{25}(X, 0)$ valuation, then the payoff at expiry from this second arbitrage strategy will be the same as for the arbitrage strategy considered in the previous paragraph.

Consequently, the Thomas approach violates the no-arbitrage principle.[2] Yet Thomas states (p. 23) that "these features [i.e. equations (A.1) to (A.4)] ensure that the no-arbitrage property is preserved." They do not.

[1] We implicitly assume away any credit risk.

[2] However, the possibility of risk-free arbitrage profits is not compatible with long-run market equilibrium. Imagine then how this situation would eventually play out. Over time, we get rich from our trading strategy. Eventually, everyone adopts our trading strategy and the whole market is selling as the price goes up and

7. Conclusions

Guy Thomas's approach to NNEG valuation based on a 'reflecting barrier' leads to considerably lower NNEG valuations than those provided by Black '76. However, his approach unravels on close scrutiny. (1) That there is something wrong with it is proven by the fact that it can deliver ERM valuations that exceed the upper bound ERM valuations implied by the PRA's ERM rational valuation Principles. (2) There is the problem that the introduction or removal of the reflecting barrier policy will likely change both the current value of the underlying price and the volatility, and Thomas does not allow for such impacts. (3) One can show that Thomas' key equation, equation (2), does not hold in the case of a deep in the money put option, which implies that that equation must itself be invalid. (4) The Thomas approach violates the no-arbitrage principle.

Since all 'reflecting barrier' option valuation models share similar basic features, it is reasonable to infer that all such models must also share the same basic flaws. In short, reflecting barrier approaches might not be as convincing as they might seem at first sight.

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buying as the price goes down, trying to make a profit. But that activity will inevitably stabilize the market price and cause volatility to collapse to low levels. This outcome, in turn, reinforces our earlier point that the adoption of a reflecting barrier policy will affect the volatility itself, i.e., it is invalid to compare valuations in a reflecting barrier world to valuations in a non-reflecting barrier world based on the same fixed volatility calibration, which is exactly what Thomas does.

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